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Determinant Properties of the Laplacian Matrix for Graphs with Self-Loops

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Abstract

Graph theory is a part of discrete mathematics that is closely connected to linear algebra. In this paper, we study the determinant of the Laplacian matrix for graphs with self-loops. We look at three types of graphs: complete graphs, complete bipartite graphs, and star graphs. For each type, we compute the determinant when there are no self-loops and when there are more.
Our main result is that if a graph has no self-loops, the Laplacian determinant is zero and the matrix is singular. But when we add at least one self-loop, the determinant becomes non-zero, and the matrix becomes non-singular. The exact value of the determinant depends on the number of self-loops and, in star graphs, on where they are placed.
However, the property of being non-singular remains the same as there is at least one self-loop. The results are the same for all three graph families. This work helps to see how self-loops change the algebraic structure of the Laplacian matrix and can be useful for future research on graphs with self-loops and complex networks.

Keywords: Complete graph, Graph, Laplacian Matrix, Self-Loop, Star graph.

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1. Introduction

Graph theory is one of the important areas of discrete mathematics. It has a close connection with linear algebra. To study the structure of a graph, we usually represent it as a matrix, such as the adjacency matrix and the Laplacian matrix. By studying the properties of these matrices, especially their determinants, we can get information about the structure of the graph. For example, we can find out whether the matrix is singular or not, whether the graph has cycles, and whether the network is stable.

So, many studies have been done on the determinant of matrices related to graphs [1] Abdullahi [1] studied the determinant of the adjacency matrix for simple graphs and showed that this value is related to the structure of the graph, such as the number of cycles. Besides the adjacency matrix, the Laplacian matrix also plays an important role in graph theory. This matrix is related to concepts such as vertex degrees, graph connectivity, and eigenvalues [2]. However, most of these studies have been done on graphs without self-loops.

In simple graphs without self-loops, the properties of the Laplacian matrix have been studied extensively, leading to important results such as the Matrix-Tree Theorem [3],[14]. This theorem connects the minors of the Laplacian matrix to the number of spanning trees of the graph. Later, this theorem was extended to directed graphs [3][4], to spanning forests [12], and to more general structures [15]. However, many real-world models and even some theoretical models include self-loops. In such cases, the structure of the Laplacian matrix changes fundamentally, making some classical results no longer directly applicable [10],[13].

In another study, Mark Jeeninga and colleagues, in their paper entitled “Graph Theoretic Formulae for the Determinant and Adjugate of Matrices Carrying Graph Structure,” have attempted to provide explicit formulae for the determinant and the adjugate matrix of matrices that carry a graph structure (such as the adjacency matrix and the Laplacian). This work is carried out with the directive to use graph properties and structure to express the determinant and adjugate, rather than performing direct algebraic computations. In this research, for matrices such as the Laplacian with in-degree and out-degree, the adjacency matrix in direct, and the Laplacian and adjacency matrices in undirected graphs, graph-based analytical relations are presented that connect the determinant and the adjugate of these matrices to the graph structure. These formulae are based on structural properties of the graph and their matrix analysis – that is, instead of a purely numerical view of the matrix, the relationship between the graph and the determinant (or adjugate) is represented in graph-oriented expressions. The methodology of this research, by treating these matrices as graph-oriented objects and using graph theory and matrix properties, derives formulae that relate the determinant and adjugate to graph parameters

such as degrees, connectivity, and the overall structure, without the need for computing all order-n products directly [5].

In another study, Deekshitha et al. [6] studied the determinant of the adjacency matrix for graphs with self-loops. In their work, self-loops were added to the vertices so that the diagonal entries of the adjacency matrix become non-zero. They studied three types of graphs: complete graphs, complete bipartite graphs, and cluster graphs. For complete graphs with self-loops the showed that the determinant of the adjacency matrix can only be one of the three values $\{-1, 0, 1\}$. For complete bipartite graphs with self-loops, the determinant is always negative. In both cases, the determinant being zero or non-zero tells us whether the graph is singular or non-singular [6].

2. Methods and materials

This research is of a theoretical-analytical nature and is based on algebraic concepts and graph theory. First, the Laplacian matrix is constructed for each graph (both loopless and with self-loops) according to standard definitions. To compute the determinant of the Laplacian matrix, depending on the matrix size, the Laplacian expansion (along rows or columns) is used, and in some cases, elementary row and column operations are applied. In certain calculations, the Schur complement is also employed for simplification. All numerical results have been verified manually and, in some cases, using Python and Mathematica software.

Loop Convention

Throughout this paper, the following convention is adopted for graphs with self-loops:

- In the adjacency matrix A, the diagonal entry corresponding to a vertex with a self-loop is set to 1, i.e., $A_{ii} = 1$. For vertices without a self-loop, $A_{ii} = 0$.
- In the degree matrix D, the degree of a vertex with a self-loop is increased by 2, since a self-loop contributes twice to the degree of its endpoint. Specifically, for a vertex i with a self-loop:
  - $\deg(v_i) = \sum_{j \neq i} A_{ij} + 2A_{ii}$
- Consequently, the Laplacian matrix is defined as $L = D - A$, and for a vertex i with a self-loop, the diagonal entry of L becomes:
  - $L_{ii} = \deg(v_i) - A_{ii} = \sum_{j \neq i} A_{ij} + A_{ii}$

This convention is consistent with the standard definitions used in Acikmese [6] and Deekshitha et al.[9].

3. Findings

3.1. Graph

A graph is a pair $G = (V, E)$ of sets such that $E \in [V]$, thus, the elements of E are 2-element subsets of V. To avoid notational ambiguities, we shall always assume tacitly that $V \cap E = \emptyset$. The elements of V are the vertices (or nodes, or points) of the graph G, the elements of E are its edges (or lines). The usual way to picture a graph is by drawing a dot for each vortices from an edge. Just how these dots and lines are drawn is considered irrelevant: all that matters is the information of which pairs of vertices from an edge and which do not [7].

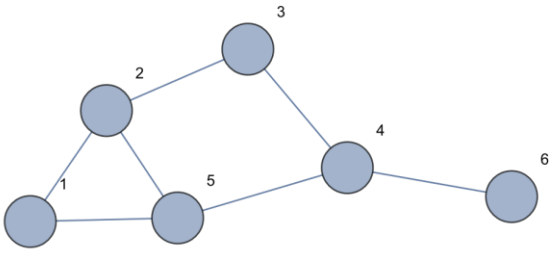


Fig. 1. Simple Graph

Loop (Self-Loop)

An edge that connects a vertex to itself is called a loop (or a self-loop) [8].

Laplacian Matrix

Let G be a simple graph with adjacency matrix A and degree matrix D. The Laplacian matrix L is defined as the difference between the degree matrix and the adjacency matrix:

$$L = D - A$$

In other words, the entries of L are given by:

$$L_{ij} := \begin{cases} \deg(v_i) & if \quad i = j \\ -1 & if\ v_i v_j \in E(G) \\ 0 & otherwise \end{cases}$$

Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$. Then $L(G) = D(G) - A(G)$, where $A(G)$ is the adjacency matrix of type $(0,1)$ and $D(G)$ is the diagonal degree matrix. The Laplacian matrix can also be expressed using the oriented incidence matrix Q as:

$$L(G) = QQ^T$$

This shows that the Laplacian matrix is independent of edge orientations. The Laplacian matrix has the following quadratic form:

$$x^T L(G)x = \sum\left(x_i - x_j\right)^2$$

Where the sum is taken over all edges $v_i, v_j \in E$. From the above relation, it follows that:

- The Laplacian matrix is symmetric:$L(G) = L(G)^T$.
- The Laplacian is positive semidefinite: $x^T L(G)x > 0$.
- The Laplacian matrix is singular.

From the definition $L(G) = D(G) - A(G)$, we have that the sum of the entries in each row of $D(G)$ equals the degree of the corresponding vertex, and the sum of the entries in each row of $A(G)$ is also equal to the same degree.

Hence, in general:

$$\sum L_{ij} = 0$$

Thus, the sum of the entries in each row of the Laplacian matrix is zero. It should be noted that the above results hold for graphs without self-loops; these properties may change [2].

Laplacian for Simple Graph

Let G be a simple graph with n vertices. The Laplacian matrix L is defined as follows:

$$L_{i,j} = \begin{cases} \deg(v_i) & , \quad i = j \\ -1 & , \quad i \neq j \ \wedge \ v_i \sim v_j \\ 0 & , \quad otherwise \end{cases}$$

Or equivalently:

$$L = D - A$$

Where D is the degree matrix, and A is the adjacency matrix. Since G is a simple graph, the entries of A consist only of 0 and 1, and all diagonal entries are zero. Now, a simple example of an undirected graph is given, along with its corresponding Laplacian matrix.

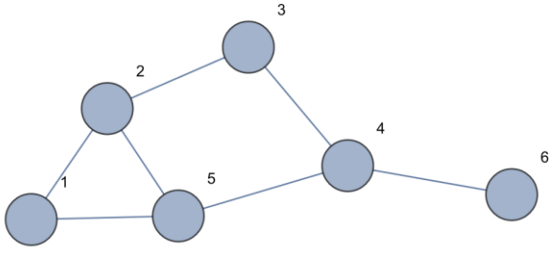


Fig. 2. A simple undirected graph with six vertices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad , \quad D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$L = D - A \Longrightarrow L = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & -1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

We observe that for the undirected graph, both the adjacency matrix and the Laplacian matrix are symmetric, and the row and column sums of the Laplacian matrix are zero (from which we directly conclude that the Laplacian matrix is singular). For directed graphs, depending on the application, either the in-degree or the out-degree may be used, as shown in the following example:

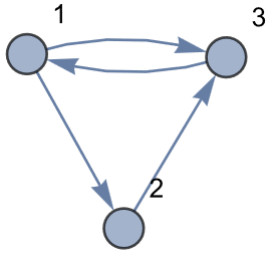


Fig. 3. Labelled graph

Table 1. The adjacency matrix and the Laplacian matrix

In-Degree Laplacian	In-Degree matrix	Out-Degree Laplacian	Out-Degree matrix	Adjacency matrix
$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

In the directed graph, the adjacency matrix and the Laplacian matrix are non-symmetric. In the Laplacian matrix, either the column sums or the row sums are zero, depending on whether the in-degree or the out-degree has been used [9].

Properties of the Laplacian Matrix

Based on the Laplacian matrix in the articles by " [2]& [10]" the following lemma expresses the properties below:

Lemma: Let $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ be an undirected graph without self-loops. Then its Laplacian matrix has the following properties:

1) $\mathrm{L(G)}$ is symmetric.
2) $\mathrm{L(G)}$ is positive semidefinite.
3) $\mathrm{L(G)}$ is singular.
4) The sum of the entries in each row of $\mathrm{L(G)}$ is zero, i.e., $\sum L_{ij} = 0$.

Proof (1):

By definition, $\mathrm{L(G)} = \mathrm{D(G)} - \mathrm{A(G)}$, where $\mathrm{D(G)}$ is the degree matrix and $\mathrm{A(G)}$ is the adjacency matrix of the graph. Since the graph is undirected, $\mathrm{A(G)}$ is symmetric, and $\mathrm{D(G)}$ is diagonal; therefore:

$$L(G)^T = D(G)^T - A(G)^T = D(G) - A(G) = L(G)$$

Hence, $\mathrm{L(G)}$ is symmetric.

Proof (2):

According to the result presented in [2]" we can write:

$$x^T L(G) x = \sum_{(i,j)\in E} \left(x_i - x_j\right)^2 \geq 0$$

Thus, the Laplacian matrix is positive semi-definite.

Proof (3):

Since we know that $\sum L_{ij} = 0$, the rows of the matrix are linearly dependent, and therefore:

$$\det\left(L(G)\right) = 0$$

Proof (4):

Again, from the definition $\mathrm{L(G)} = \mathrm{D(G)} - \mathrm{A(G)}$:

- The sum of the entries in each row of $D(G)$ equals the degree of the corresponding vertex.
- The sum of the entries in each row of $A(G)$ also equals the same degree.

Thus:

$$\sum L_{ij} = \deg(v_i) - \deg(v_i) = 0$$

Therefore, the above properties hold for graphs without self-loops. In the presence of a self-loop, however, the properties of zero row sums and singularity of the Laplacian matrix may no longer hold ([2]. [10]).

Laplacian for Graphs with Self-Loops

This section first summarizes some known facts about graph theory, and then introduces the main result of this note on graphs with self-loops. For graphs with self-loops, we will introduce the concept of pseudo-connectedness, which is useful in our developments [10].

Let $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ represent a finite graph with a set of vertices V and degree E, with $(i, j) \in E$ denoting an edge between the vertices i and j. $\mathrm{L}(\mathrm{G})$ is the Laplacian matrix for the graph G; $\alpha(\mathrm{G})$ is the algebraic connectivity of the graph G, which is the second smallest eigenvalue of $\mathrm{L}(\mathrm{G})$ [10]. E is a vertex-edge adjacency matrix. Each row of the vertex-edge adjacency matrix describes an edge between two vertices, with entries corresponding to these vertices being $+1, -1$ (it does not matter which entry is $+$ or $-$), and the rest of the entries are zeros. Note that if the edge described by a row is a self-loop, then there is only one non-zero entry with $+1$. Hence, a row of E_k, denoted by π, defining an edge between p'th and q'th vertices of the graph has its jth entry of π_j as follows:

$$\pi_j = \begin{cases} 1 & j = p \\ -1^{(q-p)} & j = q \\ 0 & 0therwise \end{cases}$$

A is the adjacency matrix, and D is the diagonal matrix of node in-degrees for G, then the following gives a relationship to compute the Laplacian matrix:

$$L(G) = E^T E = D - A \quad \ldots\ldots\ldots 1$$

The following relations are well established for a connected undirected graph G with N vertices and without any self-loops or multiple edges:

$$a(G) \geq 2\left(1 - \cos\left(\frac{\pi}{N}\right)\right) \ldots\ldots\ldots 2$$

$$2d(G) \geq max\left(\sigma\big(L(G)\big)\right) \ldots\ldots\ldots 3$$

Where $d(G)$ is the maximum in-degree of G. Indeed, the inequality is valid for any undirected graph without self-loops or multiple edges, whether it is connected or not. Also, due to the connectedness of the graph, the minimum eigenvalue of the Laplacian matrix is 0 with algebraic multiplicity of 1 and the eigenvector of 1. Next, we characterize

the location of the Laplacian eigenvalues for a connected undirected graph G with a self-loop. Having a self-loop does not change whether a graph is connected or not; that is, a graph with self-loops is connected if and only if some graph with the self-loops removed is connected. Furthermore, we define the Laplacian of an undirected graph with at least one self-loop as:

$$L(G) = L(G^0) + \sum_{(i,i)\in E} e_i {e_i}^T \quad \dots\dots\dots 4$$

Where G^0 is the largest subgraph of G with the self-loops removed, and:

$$L(G^0) = \sum_{(i,j)\in E, i\neq j} (e_i - e_j)(e_i - e_j)^T \quad \dots\dots\dots 5$$

As discussed in the lemma above, we now proceed to generalize this result to graphs containing self-loops [10].

Lemma (Laplacian in the presence of Self-Loops)

Let $G = (V, E)$ be an undirected graph, possibly with self-loops. Then the Laplacian matrix $L(G) = D(G) - A(G)$ has the following properties:

1) The Laplacian matrix is symmetric.
2) The Laplacian matrix is positive semidefinite.
3) The sum of the entries in each row of this matrix is $\sum_j L_{ij} = A_{ii}$.

Proof (1):

Since the graph is undirected, the matrix A is symmetric and D is diagonal; hence:

$$L^T = D^T - A^T = D - A = L$$

Proof (2):

For any $x \in IR^n$, we have:

$$x^T L x = x^T D x - x^T A x$$

Considering the contribution of edges and loops, we can write:

$$x^T L x = \sum_{(i,j)\in E, i\neq j} \left(x_i - x_j\right)^2 + \sum_i A_{ii} {x_i}^2 \geq 0$$

Proof (3):

From the definition of the Laplacian matrix:

$$\sum_j L_{ij} = \sum_j D_{ij} - \sum_j A_{ij}$$

Since D is diagonal:

$$\sum_j D_{ij} = \deg(v_i) = \sum_{j \neq i} A_{ij} + 2A_{ii}$$

$$\sum_j A_{ij} = \sum_{j \neq i} A_{ij} + A_{ii}$$

Therefore:

$$\sum_j L_{ij} = \left(\sum_{j \neq i} A_{ij} + 2A_{ii}\right) - \left(\sum_{j \neq i} A_{ij} + A_{ii}\right) = A_{ii}$$

Now, using "Lemma 1 from Acikmese [6]" and noting that complete graphs, complete bipartite graphs, and star graphs, when they contain at least one self-loop, are pseudo-connected, it follows that the Laplacian matrix of these graphs is positive definite, and therefore:

$$\det(L) \neq 0$$

That is, the Laplacian matrix becomes non-singular in the presence of self-loops.

Pseudo-Connectedness

An undirected graph $G = (V, E)$ with no multiple edges is called pseudo-connected if every vertex is connected to itself and/or to another vertex, and if every connected subgraph of G has at least one vertex with a self-loop [10].

According to Theorem 1 in Acikmese [6], for a pseudo-connected graph with self-loops, the Laplacian matrix is positive definite, and consequently, its determinant is non-zero.

Exact Condition for Positive Definiteness

Based on Lemma 1 and Theorem 1 of Acikmese [6], the exact condition under which the Laplacian matrix of a graph with self-loops is positive definite is as follows:

Condition: The graph G with at least one self-loop is pseudo-connected, i.e., every vertex is connected to itself and/or to another vertex, and every connected subgraph G contains at least one vertex with a self-loop.

Under this condition, Theorem 1 of Acikmese [6]states that the spectrum of the Laplacian matrix:

$$\sigma\big(L(G)\big) \subseteq \sigma + \left(L\big(\tilde{G}\big)\right) \cap [0, 2d(G^0) + 1]$$

where σ_+ denotes the set of positive eigenvalues. Consequently, all eigenvalues of $L(G)$ are strictly positive, which means the Laplacian matrix is positive definite, and therefore:

$$\det\big(L(G)\big) > 0$$

In particular, the three graph families studied in this paper-complete graphs. K_n, complete bipartite graphs $K_{m,n}$, and star graphs S_n when they contain at least one self-loop, satisfy the pseudo-connectedness condition. Therefore, for all these graphs:

$$\det(L)\ 0$$

This means the Laplacian matrix becomes non-singular in the presence of self-loops [10].

In summary, in graphs with self-loops, the properties of the Laplacian matrix change compared to the standard case. The presence of a self-loop increases the diagonal entries of the adjacency matrix, and consequently, the vertex degree-especially in terms of in-degree or out-degree also changes. Moreover, in such a case, the row sums of the Laplacian matrix are no longer necessarily zero, because the contribution of self-loops can produce non-zero values in the row sums. From a spectral point of view, the presence of self-loops also alters the Laplacian spectrum, shifting the eigenvalues and potentially affecting properties such as the zero eigenvalue and its multiplicity. As a result, self-loops not only change the degree structure and the Laplacian matrix but also lead to modifications in the spectral characteristics of the graph [10].

Theorem: Let G be a connected graph without self-loops, and let L_0 be its Laplacian matrix. Suppose we add exactly one self-loop at a vertex i, and let L be the new Laplacian matrix. Then:

$$\det(L) = \tau(G)$$

Where $\tau(G)$ is the number of spanning trees of G.

Proof: Adding one self-loop at vertex i increases the (i,i)-entry of the Laplacian matrix by 1.

Therefore,

$$L = L_0 + e_i {e_i}^T$$

By the matrix determinant lemma,

$$\det(L_0 + e_i {e_i}^T) = {e_i}^T adj(L_0) e_i$$

Since $\det(L_0) = 0$ (the graph has no self-loops), this simplifies to

$$\det(L) = {e_i}^T adj(L_0) e_i = \big(adj(L_0)\big)_{ii}$$

By the matrix-tree theorem, every cofactor of the Laplacian matrix of a connected graph equals the number of spanning trees. Hence

$$\big(adj(L_0)\big)_{ii} = \tau(G)$$

Thus,

$$\det(L) = \tau(G)$$

Corollary:

- For the complete graph K_n, using Cayley's formula $\tau(K_n) = n^{n-2}$,

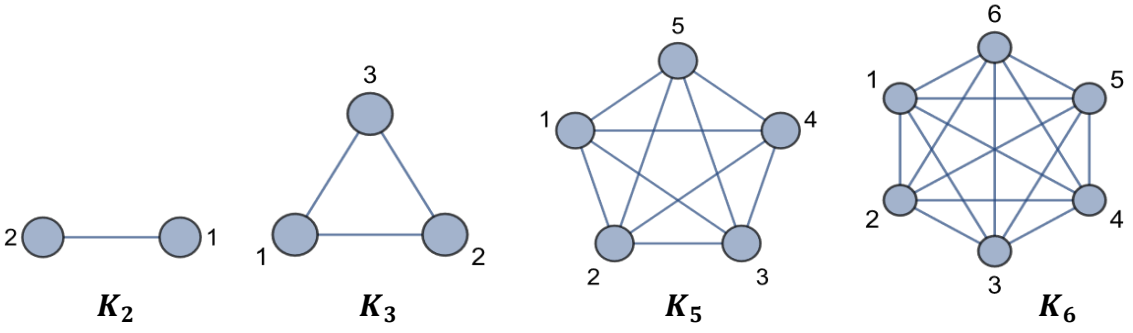
$$\det(L\ with\ one\ self-loop) = n^{n-2}$$

- For the complete bipartite graph $K_{m,n}$, using Cayley's formula $\tau(K_{m,n}) = m^{n-1}n^{m-1}$,

$$\det(L\ with\ one\ self-loop) = m^{n-1}n^{m-1}$$

Complete Graphs

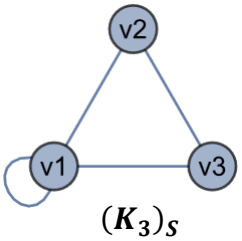
A graph G is called complete if every vertex is adjacent to all other vertices. A complete graph of order n is denoted by K_n and it is a $(n-1)$ regular graph, meaning that for every vertex $v \in (K_n), \deg(v) = n - 1$ [7].



K_2 K_3 K_5 K_6

Fig. 4. Complete Graphs

Example: Compute the determinant of the Laplacian matrix of the complete graph $(K_3)_S$ with one self-loop on vertex v_1?



$(K_3)_S$

Fig. 5. A complete graph with one self-loop

Solution:

The adjacency matrix of K_3 without loops is:

$$A(K_3) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

With one self-loop on v_1, the adjacency matrix becomes:

$$A((K_3)_S) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The degree matrix is:

$$D((K_3)_S) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Thus, the Laplacian matrix is:

$$L((K_3)_S) = D - A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Its determinant is computed as:

$$\det(L_S) = 3.\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ -1 & 2 \end{vmatrix} + (-1)\begin{vmatrix} -1 & 2 \\ -1 & -1 \end{vmatrix}$$

$$\det(L_S) = 3(4-1) + (-2-1) - (1+2)$$

$$\det(L_S) = 3$$

Thus, in the presence of a self-loop, the Laplacian determinant is non-zero, i.e., $\det(L_S) \neq 0$, meaning the Laplacian matrix becomes non-singular and invertible. In contrast, without a self-loop, the Laplacian determinant is zero, and the matrix is singular. Furthermore, for the same graph, the determinant of the adjacency matrix with a self-loop was 1, while the Laplacian determinant became 3. This indicates that the adjacency matrix and the Laplacian matrix behave differently in the presence of self-loops; that is, the determinant value changes with the addition of a self-loop.

It is also worth noting that when a graph contains self-loops, these loops play no role in counting spanning trees and do not appear in the tree structure, since a tree contains no cycles. It should be noted that increasing the number of self-loops in a complete graph does not alter the qualitative property of non-singularity, but it does change the qualitative value of the determinant.

Example: Consider the complete graph K_4 and compute its Laplacian determinant when it contains two self-loops?

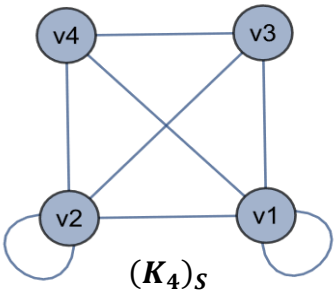


Fig. 6. A complete graph with two self-loops

Solution:

Assume the self-loops are on vertices v_1 and v_2. Then:

$$A_S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad D_S = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \quad L_S = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

$$\det(L_S) = 4\begin{vmatrix} 4 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix} + \begin{vmatrix} -1 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix} - \begin{vmatrix} -1 & 4 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 3 \end{vmatrix} + \begin{vmatrix} -1 & 4 & -1 \\ -1 & -1 & 3 \\ -1 & -1 & -1 \end{vmatrix}$$

$$\det(L_S) = 4.24 - 16 - 20 - 20 = 40$$

Thus, increasing the number of self-loops changes the value of the determinant but does not change the qualitative structure of the matrix – i.e., the Laplacian remains non-singular with self-loops.

Example (Decentralized Communication Network)

Suppose we have n radio stations, each of which can directly communicate with every other station (a fully connected network, like K_n). Occasionally, a station may need to send a message to itself (for example, a self-diagnostic command or an internal status update). This corresponds to a self-loop in graph theory.

The Laplacian matrix of this network without self-loops is singular; that is, if we attempt to solve the voltage or current equations using only the relationships between stations, the solution is not unique. However, as soon as at least one station can send a message to itself (a self-loop), the Laplacian matrix becomes non-singular (invertible).

Consequently:

- The telecommunications engineer can uniquely compute the signal power at each node.
- The network becomes more resilient to failures of some nodes.
- Routing and frequency allocation algorithms become more stable.

In summary, in a fully connected communication network, adding the ability to send a message to oneself (self-update) in at least one device makes the mathematical model of the network invertible and thus simplifies its analysis.

Example (Social Network with Influential Members)

Suppose n individuals participate in a decision-making group, where each person can directly converse with every other person (a complete graph. K_n). Each individual may also engage in self-reflection – that is, review and weigh their own opinions. This self-reflection is equivalent to a self-loop in the graph.

We now model the Laplacian matrix of this network:

- If no individual engages in self-reflection (no self-loops), the Laplacian matrix is singular. This means that the rank of the matrix is incomplete, and we cannot uniquely determine how much influence each person has on others.
- However, as soon as at least one individual practices self- reflection (a self-loop), the Laplacian matrix becomes non-singular. As a result:
    - The mathematical model of influence becomes uniquely solvable.
    - It becomes possible to precisely determine each individual's contribution to the final consensus.
    - A single individual's self- reflection is sufficient to move the entire system out of a state of linear dependence.

In summary, in complete social networks, the presence of just one individual who internally reflects on their own opinions renders the entire influence structure invertible and analyzable.

Table 2. Determinant of the Laplacian of Complete graphs with self-loops ($\det(L(K_n)_S)$)

No	No. of Vertices (n)	No. of Self-loops ($\|S\|$)	$\det(L_S)$	Logical State
1	2	0	0	Singular
2	2	1	1	Non-singular
3	2	2	3	
4	3	0	0	Singular
5	3	1	3	Non-singular
6	3	2	8	
7	3	3	16	
8	4	0	0	Singular
9	4	1	16	Non-singular
10	4	2	40	
11	4	3	75	
12	4	4	125	
13	⋮	⋮	⋮	⋮
14	n	0	0	Singular
15	n	$k \geq 1$	$\det(L_S) \neq 0$	Non-singular

In summary, as the number of self-loops increases, the determinant value changes, but the logical (qualitative) property of the matrix remains unchanged – that is, the Laplacian matrix with self-loops is always non-singular, whereas without self-loops it is singular. This result is consistent with the lemma of [10] on pseudo-connected graphs.

Bipartite Graphs

A graph G is called bipartite if its vertex set V can be partitioned into two disjoint subsets such that every edge in E has one endpoint in each of these two subsets. In other words, a graph is bipartite if its vertices can be divided into two sets V_1 and the other in V_2 [7].

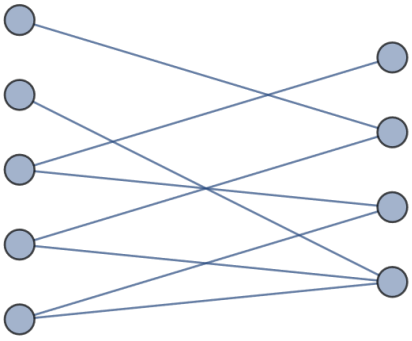


Fig. 7. A bipartite graph

Complete Bipartite Graphs

If every vertex in V_1 is connected to every vertex in V_2, the graph is called a complete bipartite graph. This graph is denoted by $K_{m,n}$, where $m = |V_1|$ and $n = |V_2|$ [7].

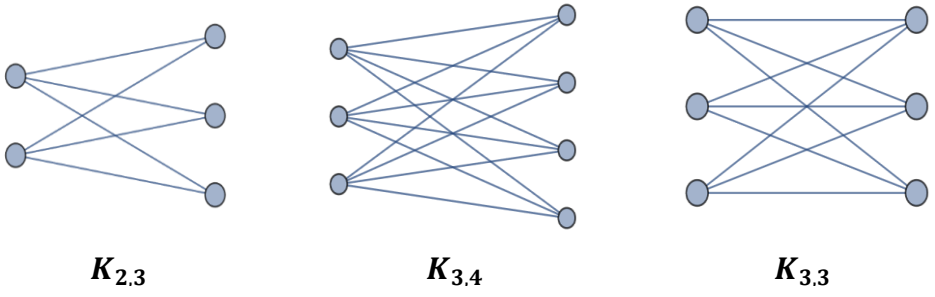


Fig. 8. Complete bipartite graphs

Example: Consider the complete bipartite graph $K_{2,3}$ with a self-loop placed on vertex a_1. Compute its Laplacian determinant?

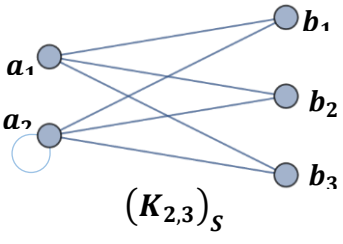


Fig. 9. The complete bipartite graph with one self-loop

Solution:

According to the standard definitions ([10]; [6]), as previously discussed:

In the adjacency matrix, the diagonal entry corresponding to the vertex with a self-loop becomes 1. And in the degree matrix, the degree of that vertex increases by 2, since a self-loop is counted twice in the degree.

Thus, the adjacency matrix, degree matrix, and Laplacian matrix are as follows:

$$A_S = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad D_S = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad L_S$$

$$= \begin{bmatrix} 4 & 0 & -1 & -1 & -1 \\ 0 & 3 & -1 & -1 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{bmatrix}$$

Using the Schur complement method on the 2×2 and 3×3 blocks, we set:

$$L_S = \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix}$$

$$P = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}, \quad Q = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}, \quad R = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 2I_3$$

Then:

$$\det(L_S) = \det(P) \cdot \det(R - Q^T P^{-1} Q) = 12 \times 1 = 12$$

Thus:

$$\det(L_S) = 12$$

We observe that the Laplacian determinant in the presence of a self-loop is non-zero, meaning the Laplacian matrix is non-singular. Therefore, a self-loop transforms the matrix from singular to non-singular and invertible. Conversely, without a self-loop, the determinant is zero, and the matrix remains singular.

Example (Online Platform for Freelancers and Employers)

Suppose there are m freelancers (specialists) and n employers (clients) registered on an online platform. Each freelancer can directly communicate with any employer (e.g., to submit a project proposal or send an offer). Freelancers do not communicate with each

other, and employers do not communicate with each other – this is precisely the structure of the complete bipartite graph. $K_{n,m}$.

Here, the vertices of the first part represent freelancers, the vertices of the second part represent employers, and the edges represent any possible work-related communication (such as proposals, negotiations, contracts, etc.).

We now analyze the Laplacian matrix:

- Without self-loops, the Laplacian matrix is singular. That is, if we try to determine the influence of each freelancer on employers or the degree of compatibility based on the existing relationships, the solution is not unique.
- If we place a self-loop on one freelancer (i.e., self-assessment), the Laplacian matrix becomes invertible. Consequently:
    - The relative competence of freelancers and the attractiveness of projects can be uniquely determined.
    - The platform can recommend the best match between a freelancer and an employer without ambiguity.
    - A single freelancer who engages in self-reflection moves the entire matching system from an incomplete state to a complete one.

In summary, in a two-sided online marketplace, adding a self-loop (self-assessment) for at least one freelancer ensures that the job recommendation algorithm operates uniquely and stably.

Example (Drug Discovery and Protein-Drug Interaction)

Suppose we have m types of pathogenic proteins and n candidate drug molecules. Each drug can directly interact with any protein (a complete bipartite graph $K_{n,m}$). Proteins do not interact with each other, and drugs do not interact with each other. In this model:

- The first part consists of proteins (disease targets).
- The second part consists of drugs (therapeutic molecules).
- Edges represent effective binding potential, as indicated by experimental data.

If we analyze the situation with self-loops, we observe that without them, the Laplacian matrix is singular – meaning the drug design model (does–effect dependency) is not unique, and optimal dosing is ambiguous.

However, if we place a self-loop on one drug, the Laplacian matrix becomes non-singular (invertible). As a result:

- The effective dose of each drug to inhibit each protein can be uniquely determined.
- Combination therapy (cocktail) optimization becomes more certain.
- A single self-binding drug (self-loop) is sufficient to transform the mathematical model of the protein- drug network from an ambiguous to a uniquely solvable state.

In summary, in intelligent drug design, the presence of at least one molecule that binds to itself (self-association effect) can make the entire interaction system analyzable and invertible.

Star Graphs

A star graph is usually denoted by S_n or $K_{1,n-1}$. It is a connected tree consisting of one central vertex (center) and a set of leaf vertices (leaves). A star graph has the following properties:

- The central vertex has degree $n-1$.
- The remaining vertices (leaves) each have degree1.
- Edges exist only between the center and the leaves.

Star graphs are also a special case of complete bipartite graphs [11].

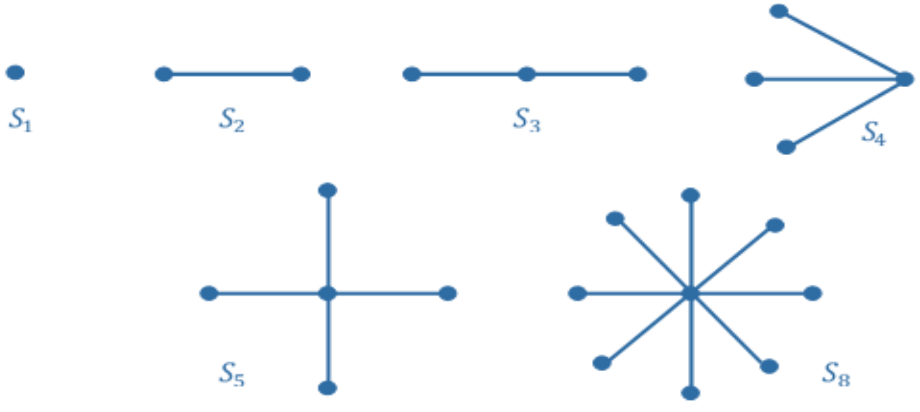


Fig. 10. Star graphs

Example: Compute the Laplacian determinant of the star graph S_4 without self-loops and with one self-loop placed on vertex v_2?

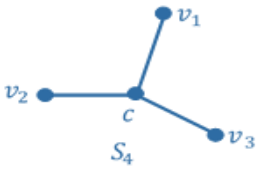


Fig. 11. Star graph with four vertices without self-loops

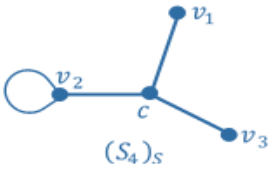


Fig. 12. Star graph with four vertices and a self-loop on a leaf

$$A(S) = \begin{matrix} \\ v_1 \\ v_2 \\ v_3 \\ c \end{matrix} \begin{matrix} \begin{matrix} v_1 & v_2 & v_3 & c \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{matrix}, \quad \det\big(A(S)\big) = 0$$

Now, we compute the adjacency determinant with a self-loop on v_2:

$$A(S)_S = \begin{matrix} \\ v_1 \\ v_2 \\ v_3 \\ c \end{matrix} \begin{matrix} \begin{matrix} v_1 & v_2 & v_3 & c \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{matrix}, \quad \det(A(S)_S) = 0$$

We observe that the adjacency determinant is zero both without and with a self-loop – hence the graph remains singular in terms of the adjacency matrix.

Now we compute the Laplacian determinant:

Without self-loops:

$$D(S) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$L(S) = D(S) - A(S) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}, \det\big(L(S)\big) = 0$$

With one self-loop on v_2:

$$A(S)_S = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad D(S)_S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$L((S)_S) = D((S)_S) - A((S)_S) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

$$\det\big(L((S)_S)\big) = \begin{vmatrix} 2 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} - 2\begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix}$$

$$= 2(3-1) - (0+1) - 2(0+1) = 4 - 1 - 2 = 1$$

Thus:

$$\det\big(L((S)_S)\big) = 1$$

We observe that the Laplacian determinant with a self-loop is non-zero, meaning the star graph with a self-loop becomes non-singular.

Example (Internal of Things – Central Node and Environmental Sensors)

Consider a central node (Hub), such as a main gateway, connected directly to $n-1$ sensors placed in various locations within a smart factory or building. Each sensor communicates only with the central node, and sensors do not communicate with each other – this is exactly the structure of a star graph. S_n.

Here, the central vertex is the main gateway, and the leaf vertices are the sensors (e.g., temperature, humidity with communication channels between the gateway and each sensor.

Now consider the case with a self-loop. Suppose a specific sensor (e.g., the temperature sensor in a server room) can self-test and send status updates to itself – for example, recording its temperature every 10 seconds and comparing it with the previous value. This self-testing corresponds to a self-loop on that leaf vertex.

Analyzing with the Laplacian matrix:

- Without a self-loop, the Laplacian matrix is singular; this means that if we try to analyze the linear dependence are not uniquely solvable, and fault detection algorithms may be ambiguous.
- With a self-loop on a sensor, the Laplacian matrix becomes non-singular. As a result:

- It can be uniquely determined whether a signal drop is due to a faulty sensor or a communication channel failure.
- Filtering and estimations (such as the Kalman filter) become more stable.
- A single self-testing sensor is sufficient to move the entire mathematical model from an incomplete to a complete state.

In summary, in industrial wireless sensor networks, adding self-testing capability to at least one sensor makes the entire system algebraically invertible, enabling unique fault detection and calibration.

Table 3. Laplacian Determinant of Star Graphs Based on the Number and Position of Self-Loops

No	No. of Vertices (n)	Self-loop configuration	$\det(L)$	Qualitative state
1	2	No self-loops	0	Singular
2	2	One self-loop (any vertex)	1	Non-singular
3	2	Two self-loops (all vertices)	3	Non-singular
4	3	No self-loops	0	Singular
5	3	One self-loop (any vertex)	1	Non-singular
6	3	Two self-loops (two leaves)	3	Non-singular
7	3	Two self-loops (two leaves)	4	Non-singular
8	3	Three self-loops (all vertices)	8	Non-singular
9	4	No self-loops	0	Singular
10	4	One self-loop (any vertex)	1	Non-singular
11	4	Two self-loops (center + one leaf)	3	Non-singular
12	4	Two self-loops (two leaves)	4	Non-singular
13	4	Four self-loops (all vertices)	20	Non-singular
14	5	No self-loops	0	Singular
15	5	One self-loop (any vertex)	1	Non-singular
16	5	Two self-loops (center + one leaf)	3	Non-singular
17	5	Two self-loops (two leaves)	4	Non-singular
18	5	Five self-loops (all vertices)	48	Non-singular
19	n	No self-loops	0	Singular
20	n	One self-loop (any vertex)	1	Non-singular

21	n	Two self-loops (center + one leaf)	3	Non-singular
22	n	Two self-loops (two leaves)	4	Non-singular
23	n	n self-loops (all vertices)	$2^{n-2}(n+1)$	Non-singular

In summary, the number and position of self-loops in star graphs affect only the quantitative value of the determinant, while the qualitative property (non-singularity) remains unchanged as long as at least one self-loop is present. The same pattern can be observed for determinants $(0,1,3,4)$ across different configurations.

4. Discussion

The results of the present study on the determinant of the Laplacian matrix with self-loops for complete graphs and star graphs reveal the following:

1. The presence of at least one self-loop in the graph renders the Laplacian matrix non-singular, i.e., $\det(L) \neq 0$. This result is obtained by invoking Lemma 1 of Acikmese [10] and the pseudo-connectedness condition of the graphs under consideration, and it is fully consistent with the viewpoint presented in that work.
2. The value of the determinant depends on the number of self-loops, and in the case of star graphs, also on their position (center versus leaf). This dependence has not been explicitly addressed in the previous literature on the Laplacian determinant. In particular, for star graphs, a fixed numerical pattern $(0,1,3,4)$ was observed for different loop configurations, which highlights the influence of the underlying tree-like structure of the graph on the determinant value and has not been reported in this form before.
3. Our study differs from the work of Deekshitha et al. [6]. They studied the determinant of the adjacency matrix for graphs with self-loops and found that for complete graphs it takes values in $\{-1,0,1\}$ In our work, we studied the Laplacian matrix instead, and we found that the determinant values can be much larger-for example, 16 for K_4 with one self-loop and 40 for K_4 with two self-loops. This shows that the adjacency matrix and the Laplacian matrix behave in fundamentally different ways when self-loops are present. So the results of one cannot be directly applied to the other.

Thus, the present study takes the first step toward providing a graph-theoretic interpretation of $(\det(L(G_S)))$ and suggests that this quantity can serve as an algebraic indicator of the distribution of self-loops and the degree pattern of the vertices.

5. Conclusion

Graph theory is one of the important branches of mathematics, and alongside it, the determinant plays a fundamental and significant role in the structure of graphs, allowing us to examine the structure of a graph. In this study, we investigated the determinant properties of the Laplacian matrix for three types of graphs: complete graphs, complete bipartite graphs, and star graphs in the presence of self-loops.

We first showed that in simple loopless graphs, the sum of the entries in each row of the Laplacian matrix is zero; hence $\det(L) = 0$, and the matrix is singular. However, by adding at least one self-loop, the row sums of the Laplacian matrix become non-zero for vertices with self-loops.

By applying Lemma 1 of [10] and noting that the graphs under consideration, when they contain a self-loop, are pseudo-connected, it follows that the Laplacian matrix in the presence of self-loops becomes non-singular and invertible, i.e. $\det(L) \neq 0$. Numerical computations also show that the determinant value depends on the number of self-loops, and in certain cases, such as star graphs, on their position as well. However, the qualitative property of non-singularity is consistently preserved as long as at least one self-loop exists.

In star graphs, a notable pattern was observed: the determinant values for configurations with no self-loop, one self-loop, and two self- loops (center + one leaf), and two self-loops (two leaves) are respective $(0,1,3,4)$. This fixed pattern, which differs from the behavior in complete graphs, indicates that the underlying tree-link structure of the star graph (prior to adding self-loops) directly influences the Laplacian determinant. This pattern is only broken when a self-loop is added to every vertex, in which case the determinant follows the general formula $2^{n-2}(n+1)$.

The findings of this research are not only of theoretical importance but are also applicable to practical problems. In various applied domains, the presence of at least one self-loop at a node makes the mathematical model invertible and enables unique analyses.

Therefore, for the three families of graphs studied, the presence of at least one self-loop is sufficient to conclude, without direct computation of the determinant, that the Laplacian matrix is non-singular. This result generalizes the classical properties of the Laplacian to non-simple graphs and can serve as a basis for future research on graphs with self-loops and complex networks.

Novelty and Comparison with Previous Work

The novelty of this study lies in the following aspects:

1. **Focus on the Laplacian determinant:** While previous works such as Acikmese [6] studied the spectrum of the Laplacian matrix for graphs with self-loops, and Deekshitha et al. [9], studied the determinant of the adjacency matrix, this paper

is the first to systematically investigate the determinant of the Laplacian matrix in the presence of self-loops.

2. **General determinant formula for one self-loop:** We prove that for any connected loopless graph G, adding exactly one self-loop yields $\det(L) = \tau(G)$, where $\tau(G)$ is the number of spanning trees. This provides a direct link between the Laplacian determinant and the number of spanning trees.
3. **Numerical pattern in star graphs:** For star graphs, we observe the fixed pattern $(0,1,3,4)$ for the determinant values under different configurations of self-loops. This pattern is not present in complete graphs and reveals the influence of the underlying tree structure on the determinant.

Comparison with previous work

Acikmese proved that the Laplacian of a pseudo-connected graph with self-loops is positive definite and thus $\det(L) \neq 0$, but did not provide explicit formulas for the determinant. Deekshitha et al. computed determinants of the adjacency matrix for graphs with self-loops and found that for complete graphs it takes values in $\{-1,0,1\}$, whereas our results show that the Laplacian determinant for complete graphs takes much larger values (e.g., 16 for K_4 with one self-loop). This confirms that the adjacency and Laplacian matrices behave fundamentally differently in the presence of self-loops.

In summary, this paper extends the classical results on the Laplacian determinant to non-simple graphs, provides a general theorem and numerical patterns that were not previously known.

Author's contributions

The author was responsible for the conceptualization, methodology, formal analysis, investigation, writing – original draft, writing – review & editing, and final approval of the manuscript.

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